Nonlinear Vibrations of Rectangular Laminated Thin Plates

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Using the basic assumption of thin-plate theory, including nonlinear terms in the von Kármán sense, the free flexural large-amplitude vibrations of arbitrarily laminated rectangular plates are studied. Full-mode displacement formulation with the aid of Galerkin's approximation is treated for more accurate analyses. The Duffing-type equation is solved by the method of harmonic balance with sufficient number of terms in time series, which is compared with the method of direct numerical integration. The convergence of the number of half waves in the x and y directions and that of time series is established. The nonlinear frequencies for the simply supported with movable edges and fully clamped with immovable edges are compared with the available solutions in the literature.

Nomenclature

$\underline{A}_{ij}$	= extensional stiffness matrix
$\bar{A}_{ij}$	= $a^2 A_{ij}/D_{11}$
a,b	= plate length in x and y directions, respectively
$\underline{B}_{ij}$	= extensional-bending coupling matrix
$\bar{B}_{ij}$	= aB_{ij}/D_{11}
$\underline{D}_{ij}$	= bending stiffness matrix
$\bar{D}_{ij}$	= D_{ij}/D_{11}
E_i	= left sides of three partial differential equations
E_L,E_T	= Young's moduli in x and y axes, respectively
f	= aspect ratio, a/b
G_{LT}	= shear modulus related to x and y axes
h	= plate thickness
J	= coefficient of inertia term in Duffing-type equation
k	= number of harmonically balancing functions
M	= resultant stress couples
N	= in-plane stress resultants
Q_{ij}	= reduced stiffness coefficients
T^0,U^0,V^0	= kinetic energy, strain energy, applied potential energy, respectively
u^0,v^0,w	= midplane deflections in x, y, and z directions, respectively
W_{11}/h	= nondimensional center point deflection by mode (1,1)
W/h	= nondimensional center point deflection (induced by all modes)
X,Y,Z	= coefficients in Duffing-type equation
X_m,Y_n	= beam eigenfunctions in x and y directions, respectively
$\varepsilon_x,\varepsilon_y,\varepsilon_{xy}$	= strains in the plate
$\varepsilon_x^0,\varepsilon_y^0,\varepsilon_{xy}^0$	= strains in midplane of plate
$\kappa_x,\kappa_y,\kappa_{xy}$	= plate curvature changes
ρ	= mass density
Ω	= ω/ω^0
$\bar{\omega}$	= ω/ω_r
ω_r	= reference frequency, $[\pi^4 D_{11}/\rho h a^4]^{1/2}$
ω^0,ω	= fundamental linear and nonlinear frequency, respectively
$(\ )_{,x}$ etc.	= partial derivative with respect to x, etc.

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Introduction

MANY papers have contributed in the understanding of the nonlinear effects of large-amplitude free vibration of thin composite plates during the last three decades. Yamaki,[1] in 1961, and Hassert and Nowinski,[2] in 1962, gave the first analyses of nonlinear vibrations of single-layer orthotropic plates. The Ritz-Galerkin technique, including thickness-shear flexibility, was used in Ref. 1 and a complicated series solution was used in Ref. 2. Wu and Vinson[3] solved the problem of nonlinear vibration of orthotropic plates using Berger's[4] approximation and Reissner's variational theorem.[5] Mayberry and Bert[6,7] performed experimental and theoretical work on linear and nonlinear vibrations of laminated plates, with all of the four edges clamped. Wu and Vinson[8] extended their earlier work[3] to symmetrically stacked laminated plates. Reissner and Stavsky[9] showed the coupling effect between bending curvatures and in-plane strains of the midplane for an unsymmetrically laminated plate in linear static analysis. Whitney and Leissa[10] presented displacement function formulation for unsymmetric angle-ply and cross-ply plates and solved the governing equation by linearization. Bennett[11] presented a nonlinear analysis for unsymmetric angle-ply laminate using stress function formulation with two terms of half-waves in the directions of plate edges. Bert[12] introduced a simplified nonlinear analysis for an arbitrarily laminated rectangular plate by assuming appropriate single mode displacement functions. A multimode (two-mode) solution for nonlinear vibration of unsymmetric all clamped and all simply supported angle-ply and cross-ply laminated plates was reported by Chandra and Raju,[13] in which stress function formulation and the perturbation technique for the resulting ordinary differential equation in time were adopted. Chia and Prabhakara[14,15] presented an analytical investigation of the nonlinear free flexural vibrations of orthotropic, cross-ply and angle-ply plates with all clamped and all simply supported edges assuming zero in-plane boundary forces. A finite element method was used by Reddy and Chao[16] to investigate the nonlinear oscillations of laminated anisotropic plate. In their displacement and slope function formulation, the time functions in the slope functions were linear and the single mode was used to make the Duffing-type equation become a standard eigenvalue equation.

A review of the literature reveals that there has not been a full multimode analysis in solving the governing equation to obtain a solution as possible in the sense of approximation technique either in stress function formulation or in that of displacements. The purpose of the present work is to deal with the nonlinear vibrations of arbitrarily laminated thin rectangular plates by taking into account a sufficient number of half-waves in assumed solution of displacement series functions and in harmonically balanced terms of time series functions to ensure the convergence and to see the effect of aspect ratio.

By using the Hamilton's variational principle, the Euler equations of motion as well as the natural and geometric boundary conditions are obtained. An assumed infinite series solution for displacements are used to get the Duffing-type equation with the aid of Galerkin's method. The Duffing-type equation is solved by the method of harmonic balance and checked by the direct numerical method of Runge-Kutta-Verner. Numerical results are shown for two out-of-plane boundary conditions and two in-plane boundary conditions and compared with some results available in the literature to show the significance of the number of terms in assumed series solutions and that of the aspect ratio of plate.

Governing Equations of Motion

Using the Kirchhoff hypothesis of classical thin plate, one can express the total strain as follows,

$$\varepsilon_x = \varepsilon_x^0 + \kappa_x z \tag{1a}$$

$$\varepsilon_y = \varepsilon_y^0 + \kappa_y z \tag{1b}$$

$$\varepsilon_{xy} = \varepsilon_{xy}^0 + \kappa_{xy} z \tag{1c}$$

Considering von Kármán-type geometric nonlinearity, one can write,

$$\varepsilon_x^0 = u_{,x}^0 + w_{,x}^2/2 \tag{2a}$$

$$\varepsilon_y^0 = v_{,y}^0 + w_{,y}^2/2 \tag{2b}$$

$$\varepsilon_{xy}^0 = u_{,y}^0 + v_{,x}^0 + w_{,x}w_{,y} \tag{2c}$$

Assuming small slopes ($w_x^2 << 1$) as well as the Kirchhoff hypothesis, one can express the midplane curvatures as follows,

$$\kappa_x = -w_{,xx} \tag{3a}$$

$$\kappa_y = -w_{,yy} \tag{3b}$$

$$\kappa_{xy} = -2w_{,xy} \tag{3c}$$

For a thin plate with an arbitrary number of anisotropic layers of arbitrary arrangements and thickness, the constitutive relations are

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix} \tag{4}$$

where N and M are the resultant forces and moments conjugate to ε^0 and κ, respectively, and A_{ij}, B_{ij}, and D_{ij} are symmetric matrices defined as follows,

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} (1, z, z^2) Q_{ij} \mathrm{d}z \qquad (i,j = 1,2,6) \tag{5}$$

Since thickness-shear flexibility can be neglected for the span-to-thickness ratio >40 (see Refs. 8 and 17), it is consistent to neglect the rotatory and coupling inertias.

Neglecting energy dissipation (i.e., damping), Hamilton's principle requires that the time integral of the difference between the potential and kinetic energies attains a stationary value, i.e.,

$$\delta \int_{t1}^{t2} (U^0 - T^0 + V^0)\, \mathrm{d}t = 0 \tag{6}$$

where δ indicates admissible variations in the degrees of freedom characterizing the state of deformation.

The strain energy is expressed as

$$U^0 = \frac{1}{2} \int_0^a \int_0^b (N_x \varepsilon_x^0 + N_y \varepsilon_y^0 + N_{xy} \varepsilon_{xy}^0 + M_x \kappa_x + M_y \kappa_y + M_{xy} \kappa_{xy})\, \mathrm{d}x\, \mathrm{d}y \tag{7}$$

and the kinetic energy is expressed as

$$T^0 = \frac{1}{2} \rho h \int_0^a \int_0^b w_{,t}^2\, \mathrm{d}x\, \mathrm{d}y \tag{8}$$

Integrating Eq. (6) by parts along with the expressions of Eqs. (2–4), (7), and (8), one can write the variational form in terms of midplane displacements as follows,

$$\begin{aligned} \delta U^0 - \delta T^0 = {} & \int_0^a \int_0^b (N_{x,x} + N_{xy,y}) \delta u^0\, \mathrm{d}x\, \mathrm{d}y \\ & + \int_0^a \int_0^b (N_{y,y} + N_{xy,x}) \delta v^0\, \mathrm{d}x\, \mathrm{d}y \\ & + \int_0^a \int_0^b (N_x w_{,xx} + N_y w_{,yy} + 2N_{xy} w_{,xy} \\ & + M_{x,xx} + M_{y,yy} + 2M_{xy,xy}) \delta w\, \mathrm{d}x\, \mathrm{d}y \\ & - \rho h \int_0^a \int_0^b w_{,t} \delta w\, \mathrm{d}x\, \mathrm{d}y - \int_0^b N_x \delta u^0 \Big|_0^a \mathrm{d}y - \int_0^a N_{xy} \delta u^0 \Big|_0^b \mathrm{d}x \\ & - \int_0^a N_y \delta v^0 \Big|_0^b \mathrm{d}x - \int_0^b N_x \delta v^0 \Big|_0^a \mathrm{d}y \\ & - \int_0^b (N_x w_{,x} + N_{xy} w_{,y} + M_{x,x} + M_{xy,y}) \delta w \Big|_0^a \mathrm{d}y \\ & - \int_0^a (N_y w_{,y} + N_{xy} w_{,x} + M_{y,y} + M_{xy,x}) \delta w \Big|_0^b \mathrm{d}x \\ & + \int_0^b M_x \delta w_{,x} \Big|_0^a \mathrm{d}y + \int_0^a M_y \delta w_{,y} \Big|_0^b \mathrm{d}x \\ & + \int_0^b M_{xy} \delta w_{,y} \Big|_0^a \mathrm{d}y + \int_0^a M_{xy} \delta w_{,x} \Big|_0^b \mathrm{d}x = 0 \end{aligned} \tag{9}$$

Now three governing partial differential equations can be written as follows (see also Ref. 18):

$$E_1(u^0, v^0, w) = L_1 u^0 + L_2 v^0 - L_3 w + w_{,x} L_1 w + w_{,y} L_2 w = 0 \tag{10a}$$

$$E_2(u^0, v^0, w) = L_2 u^0 + L_4 v^0 - L_5 w + w_{,x} L_2 w + w_{,y} L_4 w = 0 \tag{10b}$$

$$\begin{aligned} E_3(u^0, v^0, w) = {} & L_3 u^0 + L_5 v^0 - L_6 w \\ & - \rho h w_{,t} + (u_{,x}^0 + \tfrac{1}{2} w_{,x}^2) L_7 w + (v_{,y}^0 + \tfrac{1}{2} w_{,y}^2) L_8 w \\ & + (u_{,y}^0 + v_{,x}^0 + w_{,x} w_{,y}) L_9 w + w_{,x} L_3 w + w_{,y} L_5 w \\ & + 2(B_{12} - B_{66})(w_{,xy}^2 - w_{,xx} w_{,yy}) = 0 \end{aligned} \tag{10c}$$

where L_i are the linear differential operators defined by

$$L_1 = A_{11}(\ )_{,xx} + 2A_{16}(\ )_{,xy} + A_{66}(\ )_{,yy}$$

$$L_2 = A_{16}(\ )_{,xx} + (A_{12} + A_{66})(\ )_{,xy} + A_{26}(\ )_{,yy}$$

$$L_3 = B_{11}(\ )_{,xxx} + 3B_{16}(\ )_{,xxy} + (B_{12} + 2B_{66})(\ )_{,xyy} + B_{26}(\ )_{,yyy}$$

$$L_4 = A_{66}(\ )_{,xx} + 2A_{26}(\ )_{,xy} + A_{22}(\ )_{,yy}$$

$$L_5 = B_{16}(\ )_{,xxx} + (B_{12} + 2B_{66})(\ )_{,xxy} + 3B_{26}(\ )_{,xyy} + B_{22}(\ )_{,yyy}$$

$$L_6 = D_{11}(\ )_{,xxxx} + 4D_{16}(\ )_{,xxxy} + 2(D_{12} + 2D_{66})(\ )_{,xxyy} + 4D_{26}(\ )_{,xyyy} + D_{22}(\ )_{,yyyy}$$

$$L_7 = A_{11}(\ )_{,xx} + 2A_{16}(\ )_{,xy} + A_{12}(\ )_{,yy}$$

$$L_8 = A_{12}(\ )_{,xx} + 2A_{26}(\ )_{,xy} + A_{22}(\ )_{,yy}$$

$$L_9 = A_{16}(\ )_{,xx} + 2A_{66}(\ )_{,xy} + A_{26}(\ )_{,yy}$$

Boundary Conditions and Displacement Functions

To solve the three nonlinear partial differential equations simultaneously, Galerkin's method is applied in analyzing two kinds of boundary conditions.

Case 1: Simply Supported/Edge Free

Out-of-plane boundary condition: simply supported (SS)
In-plane boundary condition: edge free (EF)

$$w = M_x = N_x = N_{xy} = 0 \quad \text{at} \quad x = 0, a \tag{11a}$$

$$w = M_y = N_y = N_{xy} = 0 \quad \text{at} \quad y = 0, b \tag{11b}$$

The assumed displacement solution in the form of double Fourier series is written as

$$u^0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{mn}(t) \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \tag{12a}$$

$$v^0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_{mn}(t) \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \tag{12b}$$

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn}(t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{12c}$$

Case 2: Rigidly Clamped/Edge Fixed

Out-of-plane boundary condition: rigidly clamped (CC)
In-plane boundary condition: edge fixed (EX)

$$w = w_{,x} = u^0 = v^0 = 0 \quad \text{at} \quad x = 0, a \tag{13a}$$

$$w = w_{,y} = u^0 = v^0 = 0 \quad \text{at} \quad y = 0, b \tag{13b}$$

The assumed displacement solution in the form of double Fourier series is written as

$$u^0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{mn}(t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{14a}$$

$$v^0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_{mn}(t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{14b}$$

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn}(t) X_m(x) Y_n(y) \tag{14c}$$

where $X_m(x)$ and $Y_n(y)$ are the beam eigenfunctions, which are written as

$$X_m(x) = \cosh\alpha_m \frac{x}{a} - \cos\alpha_m \frac{x}{a} - \gamma_m\left(\sinh\alpha_m \frac{x}{a} - \sin\alpha_m \frac{x}{a}\right) \tag{15a}$$

$$Y_n(y) = \cosh\alpha_n \frac{y}{b} - \cos\alpha_n \frac{y}{b} - \gamma_n\left(\sinh\alpha_n \frac{y}{b} - \sin\alpha_n \frac{y}{b}\right) \tag{15b}$$

where α_m and γ_m are the specific coefficients for the mth flexural mode, which are calculated from the following constraints and are shown in Table 1:

$$\cosh\alpha_m \cos\alpha_m = 1 \tag{16a}$$

$$\cosh\alpha_m - \cos\alpha_m - \gamma_m(\sinh\alpha_m - \sin\alpha_m) = 0 \tag{16b}$$

Solution Method

Equations (12) and (14) are substituted into Eq. (9) and Galerkin's integrals are formally written as

$$\int_0^a \int_0^b E_1(u^0,v^0,w) \frac{\partial u^0}{\partial U_{mn}} \, dx \, dy = 0 \tag{17a}$$

$$\int_0^a \int_0^b E_2(u^0,v^0,w) \frac{\partial v^0}{\partial V_{mn}} \, dx \, dy = 0 \tag{17b}$$

$$\int_0^a \int_0^b E_3(u^0,v^0,w) \frac{\partial w}{\partial W_{mn}} \, dx \, dy = 0 \tag{17c}$$

Evaluations of the above three integrals along with Eq. (4) and the relevant boundary condition terms in Eq. (9) result in three sets of nonlinear equations in terms of time functions $\overline{U}_{mn}(\tau), \overline{V}_{mn}(\tau), \overline{W}_{mn}(\tau)$ as in Eq. (18). The coefficient matrices are written in detail in the Appendix.

$$A1_{rs}^{mn}\overline{U}_{mn} + A2_{rs}^{mn}\overline{V}_{mn} + A3_{rs}^{mn}\overline{W}_{mn} + A4_{rs}^{mnpq}\overline{W}_{mn}\overline{W}_{pq} = 0 \tag{18a}$$

$$B1_{rs}^{mn}\overline{U}_{mn} + B2_{rs}^{mn}\overline{V}_{mn} + B3_{rs}^{mn}\overline{W}_{mn} + B4_{rs}^{mnpq}\overline{W}_{mn}\overline{W}_{pq} = 0 \tag{18b}$$

$$C1_{rs}^{mn}\overline{U}_{mn} + C2_{rs}^{mn}\overline{V}_{mn} + C3_{rs}^{mn}\overline{W}_{mn} + C4_{rs}^{mnpq}\overline{W}_{mn}\overline{W}_{pq} + C5_{rs}^{mnpq}\overline{U}_{mn}\overline{W}_{pq} + C6_{rs}^{mnpq}\overline{V}_{mn}\overline{W}_{pq} + C7_{rs}^{mnpqkl}\overline{W}_{mn}\overline{W}_{pq}\overline{W}_{kl} + J_{rs}^{mn}\overline{W}_{mn,\tau\tau} = 0 \tag{18c}$$

Table 1 Values of α_m and γ_m

m	α_m	γ_m
1	4.7300407448627040 30	0.98250221457623807
2	7.8532046240958375 57	1.00077731190726905
3	10.99560783800167100	0.99996645012540900
4	14.13716549125746410	1.00000144989765650
5	17.27875965739948100	0.99999993734438300
6	20.42035224562606100	1.00000000270759500
$m > 6$	$(2m + 1)\pi/2$	1.0

Table 2 Elastic constants

Material	E_L/E_T	G_{LT}/E_T	ν_{LT}
Graphite/epoxy	40	0.5	0.25
Boron/epoxy	10	1/3	0.3
Carbon fiber reinforced plastic	7.6	0.41	0.3
Aluminum	1.0	0.3846	0.3

The following nondimensionalizing parameters are used in getting Eqs. (18)

$$\bar{U}_{mn} \equiv U_{mn}/h, \qquad \bar{V}_{mn} \equiv V_{mn}/h, \qquad \bar{W}_{mn} \equiv W_{mn}/h$$
$$\xi \equiv x/a, \qquad \eta \equiv y/b, \qquad \tau \equiv t\omega_r \qquad (19)$$

Algebraically computing the vectors $\bar{U}_{mn}(\tau)$ and $\bar{V}_{mn}(\tau)$ from Eqs. (18a) and (18b) and substituting them into Eq. (18c) gives the Duffing-type equation in the tensorial form as

$$J_{rs}^{mn}\frac{d^2\bar{W}_{mn}}{d\tau^2} + X_{rs}^{mn}\bar{W}_{mn} + Y_{rs}^{mnpq}\bar{W}_{mn}\bar{W}_{pq} + Z_{rs}^{mnpqkl}\bar{W}_{mn}\bar{W}_{pq}\bar{W}_{kl} = 0 \qquad (20)$$

Since an exact solution of Eq. (20) is not known, an approximate solution is obtained by the method of harmonic balance (MHB). The deflection coefficients $\bar{W}_{mn}(\tau)$ are expanded into Fourier cosine series in τ as

$$\bar{W}_{mn}(\tau) = \sum_{k=1}^{\infty} A_{mn}^{(k)} \cos k\bar{\omega}\tau \qquad (21)$$

where $A_{mn}^{(k)}$ are the constant Fourier coefficients. This assumed solution is inserted into Eq. (20) and each term is converted into the first power of cosine functions. Equating the coefficients of like terms of cosine to zero, a system of simultaneous nonlinear algebraic equations are obtained. These equations are solved for $A_{mn}^{(k)}$ by a modified Newton-Raphson algorithm.

Results and Discussion

There are so many parameters that can be varied in the nonlinear vibration of anisotropic plates that it would be difficult to present and compare results for all cases. Only a few typical cases will be selected for discussion. Four kinds of materials are used for comparison with available results in the literature. The material properties are shown in Table 2.

The results of a linear analysis that showed excellent agreement for SS/EF boundary condition[14,15] and for CC/EX boundary condition[12,19] are not shown in this paper. With the present formulation involving an arbitrary number of terms it enables one to truncate the series to establish convergence. Since two kinds of series solutions are used here in spatial and time functions, a sufficient number of terms in each series solution are to be investigated first. Figures 1 and 2 show the required harmonically balancing terms for sufficient convergence in nonlinear analysis for two different boundary conditions. In both cases, two terms in the x and y directions of Fourier displacement series are used for two-layer [±45] angle-ply square graphite/epoxy plate. The SS/EF case is less sensitive to the number of harmonically balancing terms than CC/EX case. It is seen that a one-term MHB can be used with sufficient accuracy in the SS/EF case, especially for isotropic or orthotropic plates. In both boundary conditions, it can be seen that no more than three terms in MHB are necessary for convergence. Figures 3 and 4 show the convergence on the number of half-waves in the x and y directions for SS/EF and CC/EX cases, respectively. In both cases, the ratio of fundamental nonlinear to linear frequency is 1.3 for two-layer [±45] angle-ply square graphite/epoxy plate. They show that at least three terms in displacement series in both the x and y directions are essential for an accurate analysis and the one-term analysis, which was tried in many works in the past, is somewhat conservative, as can be seen in Figs. 5–7. In Figs. 5 and 6, the importance of the number of half-waves is stressed.

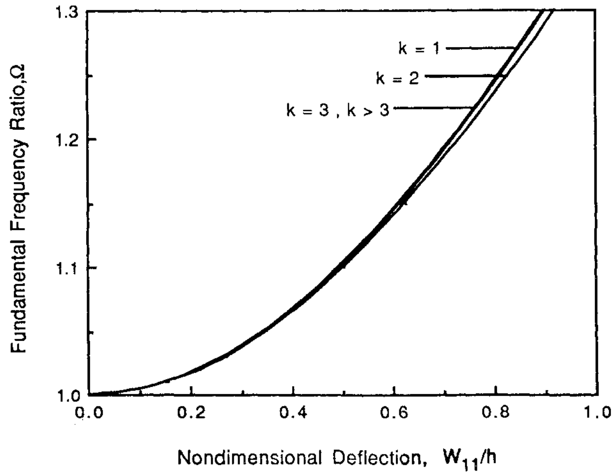

Fig. 1 Effect of number of harmonically balancing functions on convergence (SS/EF, two-layer [±45] angle-ply square graphite/epoxy plate).

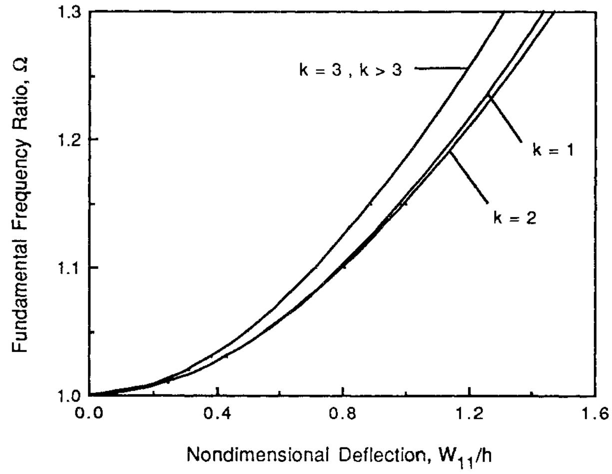

Fig. 2 Effect of number of harmonically balancing functions on convergence (CC/EX, two-layer [±45] angle-ply square graphite/epoxy plate).

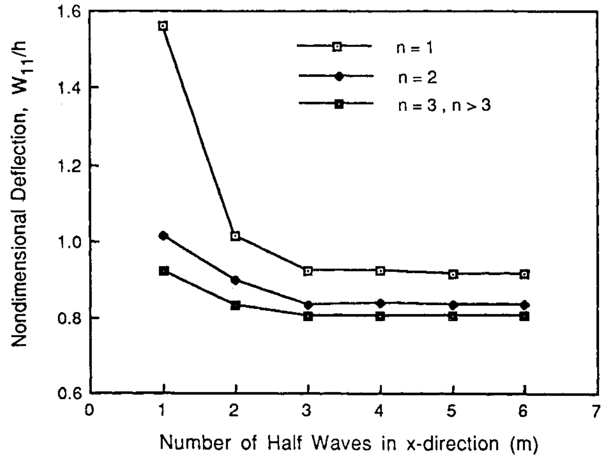

Fig. 3 Effect of number of half-waves on convergence (SS/EF, two-layer [±45] angle-ply square graphite/epoxy plate, Ω = 1.3).

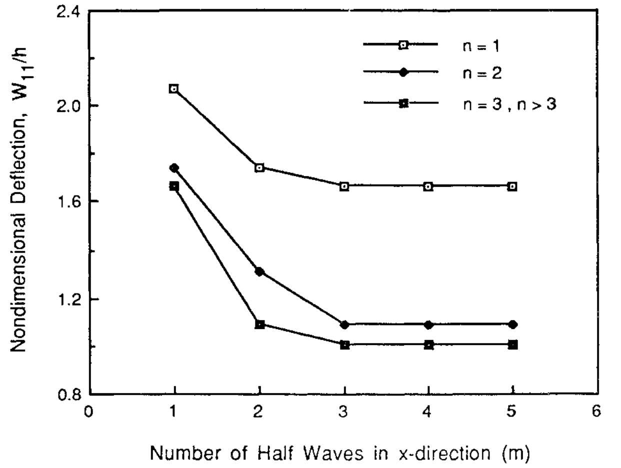

Fig. 4 Effect of number of half-waves on convergence (CC/EX, two-layer [±45] angle-ply square graphite/epoxy plate, Ω = 1.3).

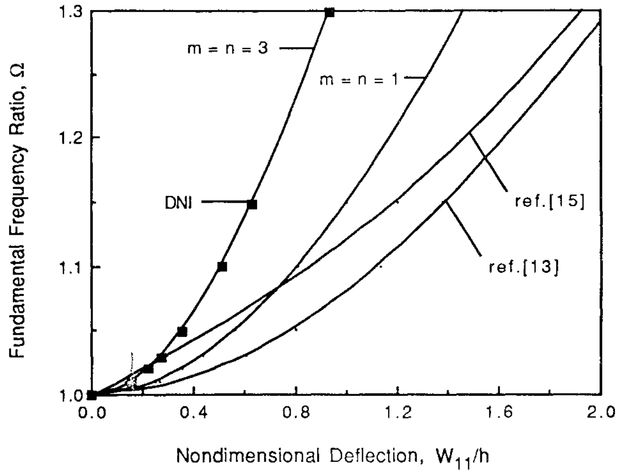

Fig. 5 Comparison of fundamental frequency ratio (SS/EF, two-layer [±45] angle-ply square CFRP plate); DNI indicates solution using direct numerical integration.

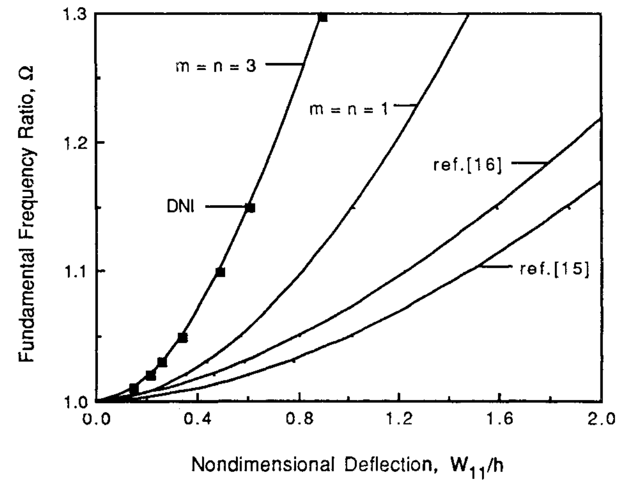

Fig. 6 Comparison of fundamental frequency ratio (SS/EF, two-layer [±45] angle-ply square Boron/Epoxy plate); DNI indicates solution using direct numerical integration.

The calculations are carried out using two-layer [±45] angle-ply square carbon fiber reinforced plastic (CFRP) and Boron/Epoxy plates in Figs. 5 and 6, respectively. The two referenced curves in Fig. 5 were based on one-term analysis for SS/EF case and are redrawn using the plots in Refs. 13 and 15. The curve of Ref. 16 in Fig. 6 was generated by a finite element method for nonlinear analysis, but linearized to get a standard eigenvalue equation. It is noted that the subject formulation gives much higher fundamental frequency ratio at certain plate deflection and it is believed to be caused by different mathematical formulation.

It is known that the displacement function formulation gives upper bounds for the exact frequencies, whereas the mixed formulation of stress function and vertical deflection function does not have any bounding property. In these figures, the nine-term MHB are compared with the direct numerical integration method of sixth-order Runge-Kutta-Verner to obtain excellent agreements with each other.

If we want to see the effects of aspect ratios in nonlinear analyses it is again emphasized to use a sufficient number of terms in an assumed series solution, as shown in Fig. 7. The calculations are for the two-ply [±45] angle-ply square graphite/epoxy plate with SS/EF condition. It is seen from Fig. 7 that not only the linear fundamental frequency but also the fundamental frequency ratio for a given deflection decreases with the aspect ratio in multimode analysis. This trend in the fundamental frequency ratio is in reverse to that of the linear fundamental frequency in single-mode analysis given in Ref. 15.

The totally clamped case (CC/EX) has not been dealt with widely. Figure 8 shows some analytical and experimental results for a square isotropic plate with the Poisson ratio ν of 0.3. The Duffing-type equation derived from a single-term

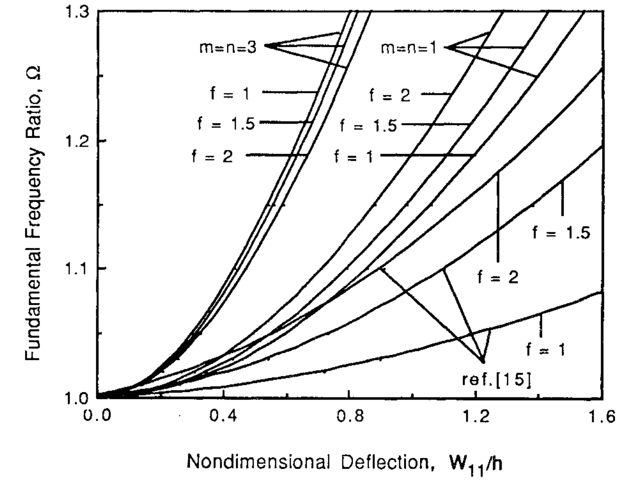

Fig. 7 Effect of aspect ratio on fundamental frequency ratio (SS/EF, two-layer [±45] angle-ply Graphite/Epoxy plate).

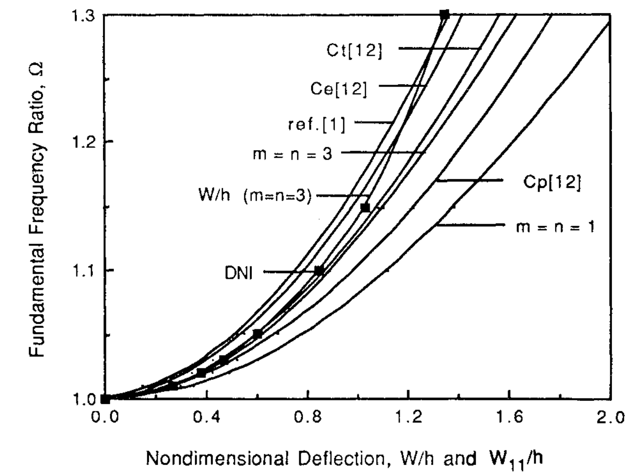

Fig. 8 Comparison of fundamental frequency ratio (CC/EX, square isotropic plate, ν = 0.3); DNI indicates solution using direct numerical integration.

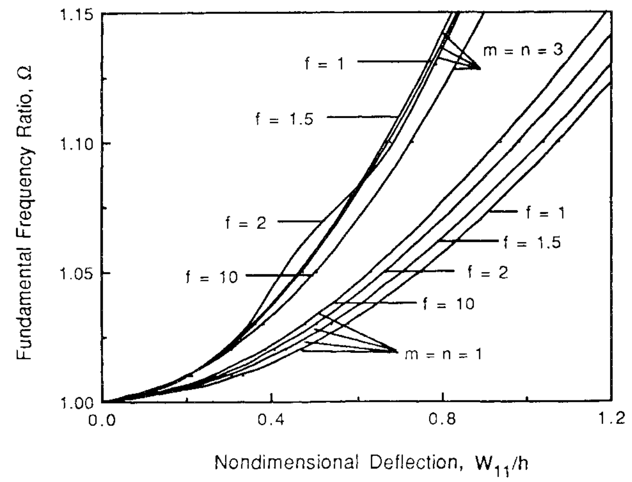

Fig. 9 Effect of aspect ratio on fundamental frequency ratio (CC/EX, two-layer cross-ply CFRP plate).

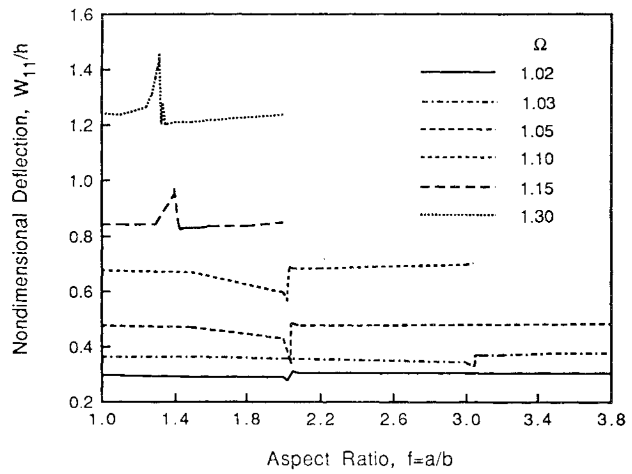

Fig. 10 Effect of aspect ratio on plate deflection for various fundamental frequency ratios (CC/EX, two-layer cross-ply CFRP plate).

mixed formulation was solved by assuming time function as elliptic cosine in Ref. 1. Even though the edge-parallel movement (i.e., $N_{xy} = 0$) is allowed for the all clamped plate, the results do not differ significantly with the results of the present work. An experimental work[6] done for an anisotropic plate having the aspect ratio of 1.5 was converted to a square plate in Ref. 12 (curve Ce[12]). Single-term polynomial-type and trigonometric-type displacement functions were tried in Ref. 12 for nonlinear large-amplitude vibration (curves Cp and Ct[12]). The results using a trigonometric-type displacement function in Ref. 12 agree well with the present nine-term analysis for lower Ω, which is also checked by direct numerical integration. In this case, the nondimensional center point deflection involves the effect of total modes for comparison. The fundamental mode deflection is also shown in this figure.

Finally, the effect of aspect ratio for fully clamped two-layer cross-ply CFRP plate are shown in Figs. 9 and 10. It is noted that the linear fundamental frequency always decreases with the aspect ratio, whereas the fundamental frequency ratio for a given deflection increases but has a limit corresponding to a specific fundamental frequency ratio. The curve for the aspect ratio of 2 in Fig. 9 reveals the jump of deflection of fundamental mode in the higher fundamental frequency ratios. The jump of deflection occurs at the so-called critical aspect ratio and transfers to the lower fundamental frequency ratios with the aspect ratio, as can be seen in Fig. 10. It is also noted that the jump size of the plate deflection, which indicates the energy transfer to and from the fundamental mode, is getting bigger for the higher fundamental frequency ratio due to the large magnitude of linear fundamental frequency in the small aspect ratio. The one-term solutions in Fig. 9 do not show the aforementioned phenomenon as the aspect ratio varies.

Conclusions

Two quite different boundary conditions (SS/EF and CC/EX) have been analyzed by utilizing Galerkin's method, Fourier double series displacement functions with the aid of the method of harmonic balance, and direct numerical integration. It has been shown that a sufficient number of terms in assumed series solution is essential in the analysis of nonlinear, large-amplitude oscillation of plates. The sufficient number of terms in double Fourier displacement series is found to be three for both SS/EF and CC/EX cases for square plates. It is also shown that three terms in harmonically balancing time series gives sufficient convergence. For the case of a simply supported angle-ply plate, there is appreciable quantitative differences in the frequencies due to the number of half-waves in the assumed displacement series solutions. It is noted that the fundamental frequency ratio decreases with the aspect ratio multimode analysis for a given deflection but the trend is reversed for single-mode analysis in the case of SS/EF. In the case of a fully clamped plate (CC/EX), the fundamental frequency ratio increases first with the aspect ratio for a given deflection but decreases again beginning from the high fundamental frequency ratio point as the aspect ratio gets larger, and the plate deflection keeps increasing for sufficiently large value of aspect ratio for a given fundamental frequency ratio.

Appendix

All of the coefficient matrices in Eqs. (18) are shown in the following equations for two different boundary conditions with some notational definitions.

Case 1: Simply Supported/Edge Free

$$S_m \equiv \sin m\pi\xi, \qquad C_m \equiv \cos m\pi\xi$$

$$\bar{A}_{mr} \equiv \int_0^1 S_m S_r \, d\xi, \quad \bar{B}_{mr} \equiv \int_0^1 S_m C_r \, d\xi, \quad \bar{C}_{mr} \equiv \int_0^1 C_m C_r \, d\xi$$

$$\bar{G}_{mpr} \equiv \int_0^1 S_m S_p S_r \, d\xi, \qquad \bar{H}_{mpr} \equiv \int_0^1 S_m S_p C_r \, d\xi$$

$$\bar{I}_{mpr} \equiv \int_0^1 S_m C_p C_r \, d\xi, \qquad \bar{J}_{mpr} \equiv \int_0^1 C_m C_p C_r \, d\xi$$

$$\bar{K}_{mpkr} \equiv \int_0^1 S_m S_p S_k S_r \, d\xi, \qquad \bar{L}_{mpkr} \equiv \int_0^1 S_m S_p S_k C_r \, d\xi$$

$$\bar{M}_{mpkr} \equiv \int_0^1 S_m S_p C_k C_r \, d\xi, \qquad \bar{N}_{mpkr} \equiv \int_0^1 S_m C_p C_k C_r \, d\xi$$

$$\bar{E}_{mr} \equiv 1 - (-1)^{m+r}, \qquad \bar{F}_{mpr} \equiv 1 - (-1)^{m+p+r}$$

The coefficient matrices of Eqs. (18) are written below as,

$$A1_{rs}^{mn} = (\bar{A}_{11}m^2 + \bar{A}_{66}f^2n^2)\bar{C}_{mr}\bar{C}_{ns} - 2\bar{A}_{16}fmn\,\bar{B}_{mr}\bar{B}_{ns} + \bar{A}_{16}\frac{fn}{\pi}\bar{E}_{mr}\bar{B}_{ns} + \bar{A}_{16}\frac{fm}{\pi}\bar{E}_{ns}\bar{B}_{mr} \tag{A1}$$

$$A2_{rs}^{mn} = (\bar{A}_{16}m^2 + \bar{A}_{26}f^2n^2)\bar{C}_{mr}\bar{C}_{ns} - (\bar{A}_{12} + \bar{A}_{66})fmn\,\bar{B}_{mr}\bar{B}_{ns} + \bar{A}_{12}\frac{fn}{\pi}\bar{E}_{mr}\bar{B}_{ns}$$

$$+ \overline{A}_{66}\frac{fm}{\pi}\overline{E}_{ns}\overline{B}_{mr} \tag{A2}$$

$$\begin{aligned} A3_{rs}^{mn} = & -\pi[\overline{B}_{11}m^3 + (\overline{B}_{12} + 2\overline{B}_{66})f^2mn^2]\overline{C}_{mr}\overline{B}_{ns} \\ & - \pi(3\overline{B}_{16}fm^2n + \overline{B}_{26}f^3n^3)\overline{B}_{mr}\overline{C}_{ns} \\ & + 2\overline{B}_{16}fmn\overline{E}_{mr}\overline{C}_{ns} + 2\overline{B}_{66}f^2mn\overline{E}_{ns}\overline{C}_{mr} \end{aligned} \tag{A3}$$

$$\begin{aligned} A4_{rs}^{mnpq} = & \frac{h}{a}\{\pi f(\overline{A}_{16}m^2q + \overline{A}_{26}f^2n^2q)\overline{H}_{mpr}\overline{I}_{nqs} \\ & - \pi f^2[(\overline{A}_{12} + \overline{A}_{66})mnq\overline{J}_{nqs} - \overline{A}_{66}mq^2\overline{H}_{nqs}]\overline{I}_{pmr} \\ & + \overline{A}_{11}m^2p\pi\overline{I}_{mpr}\overline{H}_{nqs} - 2\overline{A}_{16}fmnp\pi\overline{J}_{mpr}\overline{I}_{qns} \\ & - \tfrac{1}{2}\overline{A}_{11}mp\overline{F}_{mpr}\overline{H}_{nqs} - \tfrac{1}{2}\overline{A}_{26}f^3nq\overline{F}_{nqs}\overline{H}_{mpr}\} \end{aligned} \tag{A4}$$

$$\begin{aligned} B1_{rs}^{mn} = & (\overline{A}_{16}m^2 + \overline{A}_{26}f^2n^2)\overline{C}_{mr}\overline{C}_{ns} \\ & - (\overline{A}_{12} + \overline{A}_{66})fmn\overline{B}_{mr}\overline{B}_{ns} + \overline{A}_{12}\frac{fm}{\pi}\overline{E}_{ns}\overline{B}_{mr} \\ & + \overline{A}_{66}\frac{fn}{\pi}\overline{E}_{mr}\overline{B}_{ns} \end{aligned} \tag{A5}$$

$$\begin{aligned} B2_{rs}^{mn} = & (\overline{A}_{66}m^2 + \overline{A}_{22}f^2n^2)\overline{C}_{mr}\overline{C}_{ns} - 2\overline{A}_{26}fmn\overline{B}_{mr}\overline{B}_{ns} \\ & + \overline{A}_{26}\frac{fm}{\pi}\overline{E}_{ns}\overline{B}_{mr} + \overline{A}_{26}\frac{fn}{\pi}\overline{E}_{mr}\overline{B}_{ns} \end{aligned} \tag{A6}$$

$$\begin{aligned} B3_{rs}^{mn} = & -\pi(\overline{B}_{16}m^3 + 3\overline{B}_{26}f^2mn^2)\overline{C}_{mr}\overline{B}_{ns} \\ & - \pi[(\overline{B}_{12} + 2\overline{B}_{66})fm^2n + \overline{B}_{22}f^3n^3]\overline{B}_{mr}\overline{C}_{ns} \\ & + 2\overline{B}_{26}f^2mn\,\overline{E}_{ns}\overline{C}_{mr} + 2\overline{B}_{66}fmn\,\overline{E}_{mr}\overline{C}_{ns} \end{aligned} \tag{A7}$$

$$\begin{aligned} B4_{rs}^{mnpq} = & \frac{h}{a}[\pi f(\overline{A}_{66}m^2q + \overline{A}_{22}f^2n^2q)\overline{H}_{mpr}\overline{I}_{nqs} \\ & - \pi f^2(2\overline{A}_{26}mnq\overline{J}_{nqs} - \overline{A}_{26}mq^2\overline{H}_{nqs})\overline{I}_{pmr} \\ & + \overline{A}_{16}m^2p\pi\overline{I}_{mpr}\overline{H}_{nqs} - (\overline{A}_{12} + \overline{A}_{66})fmnp\pi\,\overline{J}_{mpr}\overline{I}_{qns} \\ & - \tfrac{1}{2}\overline{A}_{22}f^3nq\overline{F}_{nqs}\overline{H}_{mpr} - \tfrac{1}{2}\overline{A}_{16}mp\,\overline{F}_{mpr}\overline{H}_{nqs}] \end{aligned} \tag{A8}$$

$$\begin{aligned} C1_{rs}^{mn} = & -\frac{4}{\pi}\Big\{[\overline{B}_{11}m^3 + (\overline{B}_{12} + 2\overline{B}_{66})f^2mn^2]\overline{A}_{mr}\overline{B}_{sn} \\ & + (3\overline{B}_{16}fm^2n + \overline{B}_{26}f^3n^3)\overline{B}_{rm}\overline{A}_{sn} \\ & + \frac{1}{\pi}\overline{B}_{16}fnr\overline{E}_{mr}\overline{A}_{ns} + \frac{1}{\pi}\overline{B}_{12}f^2ms\overline{E}_{ns}\overline{A}_{mr}\Big\} \end{aligned} \tag{A9}$$

$$\begin{aligned} C2_{rs}^{mn} = & -\frac{4}{\pi}\Big\{[(\overline{B}_{12} + 2\overline{B}_{66})fm^2n + \overline{B}_{22}f^3n^3]\overline{B}_{rm}\overline{A}_{sn} \\ & + (\overline{B}_{16}m^3 + 3\overline{B}_{26}f^2mn^2)\overline{A}_{mr}\overline{B}_{sn} + \frac{1}{\pi}\overline{B}_{12}fnr\,\overline{E}_{mr}\overline{A}_{ns} \\ & + \frac{1}{\pi}\overline{B}_{26}f^2ms\,\overline{E}_{ns}\overline{A}_{mr}\Big\} \end{aligned} \tag{A10}$$

$$\begin{aligned} C3_{rs}^{mn} = & -4\Big\{-[m^4 + 2(\overline{D}_{12} + 2\overline{D}_{66})f^2m^2n^2 \\ & + \overline{D}_{22}f^4n^4]\overline{A}_{mr}\overline{A}_{ns} + 4(\overline{D}_{16}fm^3n + \overline{D}_{26}f^3mn^3)\overline{B}_{rm}\,\overline{B}_{sn} \\ & + \frac{2}{\pi}\overline{D}_{16}fmnr\overline{E}_{mr}\overline{B}_{sn} + \frac{2}{\pi}\overline{D}_{26}f^3mns\,\overline{E}_{ns}\overline{B}_{rm}\Big\} \end{aligned} \tag{A11}$$

$$\begin{aligned} C4_{rs}^{mnpq} = & -\frac{4h}{\pi a}[(\overline{A}_{11}mp^2 + \overline{A}_{12}f^2mq^2)\overline{G}_{mpr}\overline{H}_{qsn} \\ & + f(\overline{A}_{16}np^2 + \overline{A}_{26}f^2nq^2)\overline{H}_{prm}\overline{G}_{nqs} - 2\overline{A}_{16}fmpq\overline{H}_{mrp}\overline{I}_{snq} \\ & - 2\overline{A}_{66}f^2npq\overline{I}_{rmp}\overline{H}_{nsq}] \end{aligned} \tag{A12}$$

$$\begin{aligned} C5_{rs}^{mnpq} = & -\frac{4h}{\pi a}[(\overline{A}_{16}mp^2 + \overline{A}_{26}f^2mq^2)\overline{G}_{mpr}\overline{H}_{qsn} \\ & + f(\overline{A}_{12}np^2 + \overline{A}_{22}f^2nq^2)\overline{H}_{prm}\overline{G}_{nqs} \\ & - 2\overline{A}_{26}f^2npq\overline{I}_{rmp}\overline{H}_{nsq} - 2\overline{A}_{66}fmpq\overline{H}_{mrp}\overline{I}_{snq}] \end{aligned} \tag{A13}$$

$$\begin{aligned} C6_{rs}^{mnpq} = & -\frac{4h}{a}\{[2(\overline{B}_{12} - \overline{B}_{66})f^2mnpq\overline{I}_{snq} \\ & - \overline{B}_{11}m^3p\overline{G}_{nqs}]\overline{I}_{rmp} - f(\overline{B}_{16}m^3q + 3\overline{B}_{26}f^2mn^2q)\overline{H}_{prm}\overline{H}_{nsq} \\ & - f^2[(\overline{B}_{12} + 2\overline{B}_{66})m^2nq + \overline{B}_{22}f^2nq^3]\overline{G}_{mpr}\overline{I}_{snq} \\ & - f^2[(\overline{B}_{12} + 2\overline{B}_{66})mn^2p\overline{I}_{rmp} \\ & + 2(\overline{B}_{12} - \overline{B}_{66})m^2q^2\overline{G}_{mpr}]\overline{G}_{nqs} \\ & - 3\overline{B}_{16}fm^2np\overline{H}_{mrp}\overline{H}_{qsn} - \overline{B}_{26}f^3mq^3\overline{H}_{prm}\overline{H}_{nsq} \\ & - \frac{1}{2\pi}\overline{B}_{11}mpr\,\overline{F}_{mpr}\overline{G}_{nqs} - \frac{1}{2\pi}\overline{B}_{22}f^4nqs\overline{F}_{nqs}\overline{G}_{mpr}\} \end{aligned} \tag{A14}$$

$$\begin{aligned} C7_{rs}^{mnpqkl} = & -4\frac{h^2}{a^2}[-\tfrac{1}{2}(\overline{A}_{11}m^2pk + \overline{A}_{12}f^2n^2pk)\overline{M}_{mrpk}\overline{K}_{nqls} \\ & - \tfrac{1}{2}f^2(\overline{A}_{12}m^2ql + \overline{A}_{22}f^2n^2ql)\overline{K}_{mpkr}\overline{M}_{nsql} - f(\overline{A}_{16}m^2pl \\ & + \overline{A}_{26}f^2n^2pl)\overline{L}_{mkrp}\overline{L}_{nqsl} + \overline{A}_{16}fmnpk\,\overline{N}_{rmpk}\overline{L}_{qlsn} \\ & + \overline{A}_{26}f^3mnql\,\overline{L}_{pkrm}\overline{N}_{snql} \\ & + 2\overline{A}_{66}f^2mnpl\,\overline{M}_{krmp}\overline{M}_{qsnl}] \end{aligned} \tag{A15}$$

$$J_{rs}^{mn} = 4\overline{A}_{mr}\overline{A}_{ns} \tag{A16}$$

Case 2: Rigidly Clamped/Edge Fixed

$$\overline{E}1_{mr} \equiv \int_0^1 X_mS_r\,d\xi, \qquad \overline{E}2_{mr} \equiv \int_0^1 X_{m,\xi}S_r\,d\xi$$

$$\overline{E}3_{mr} \equiv \int_0^1 X_{m,\xi\xi}S_r\,d\xi, \qquad \overline{E}4_{mr} \equiv \int_0^1 X_{m,\xi\xi\xi}S_r\,d\xi$$

$$\overline{F}1_{mr} \equiv \int_0^1 X_mC_r\,d\xi, \qquad \overline{P}1_{mr} \equiv \int_0^1 X_mX_r\,d\xi$$

$$\overline{P}2_{mr} \equiv \int_0^1 X_{m,\xi}X_r\,d\xi, \qquad \overline{P}3_{mr} \equiv \int_0^1 X_{m,\xi\xi}X_r\,d\xi$$

$$\overline{P}4_{mr} \equiv \int_0^1 X_{m,\xi\xi\xi}X_r\,d\xi, \qquad \overline{P}5_{mr} \equiv \int_0^1 X_{m,\xi\xi\xi}X_r\,d\xi$$

$$\overline{G}1_{mpr} \equiv \int_0^1 X_mX_pS_r\,d\xi, \qquad \overline{G}2_{mpr} \equiv \int_0^1 X_{m,\xi}X_pS_r\,d\xi$$

$$\overline{G}3_{mpr} \equiv \int_0^1 X_{m,\xi\xi}X_pS_r\,d\xi, \qquad \overline{G}4_{mpr} \equiv \int_0^1 X_{m,\xi}X_{p,\xi}S_r\,d\xi$$

$$\overline{G}5_{mpr} \equiv \int_0^1 X_{m,\xi\xi}X_{p,\xi}S_r\, d\xi, \quad \overline{H}1_{mpr} \equiv \int_0^1 X_m X_p C_r\, d\xi$$

$$\overline{H}2_{mpr} \equiv \int_0^1 X_{m,\xi}X_p C_r\, d\xi, \quad \overline{H}3_{mpr} \equiv \int_0^1 X_{m,\xi\xi}X_p C_r\, d\xi$$

$$\overline{Q}1_{mpk} \equiv \int_0^1 X_m X_p X_k\, d\xi, \quad \overline{Q}2_{mpk} \equiv \int_0^1 X_{m,\xi}X_p X_k\, d\xi$$

$$\overline{Q}3_{mpk} \equiv \int_0^1 X_{m,\xi}X_{p,\xi}X_k\, d\xi, \quad \overline{Q}4_{mpk} \equiv \int_0^1 X_{m,\xi\xi}X_p X_k\, d\xi$$

$$\overline{Q}5_{mpk} \equiv \int_0^1 X_{m,\xi\xi}X_{p,\xi}X_k\, d\xi, \quad \overline{Q}6_{mpk} \equiv \int_0^1 X_{m,\xi\xi\xi}X_p X_k\, d\xi$$

$$\overline{Q}7_{mpk} \equiv \int_0^1 X_{m,\xi\xi\xi}X_{p,\xi}X_k\, d\xi, \quad \overline{R}1_{mpkl} \equiv \int_0^1 X_m X_p X_k X_l\, d\xi$$

$$\overline{R}2_{mpkl} \equiv \int_0^1 X_{m,\xi}X_p X_k X_l\, d\xi, \quad \overline{R}3_{mpkl} \equiv \int_0^1 X_{m,\xi}X_{p,\xi}X_k X_l\, d\xi$$

$$\overline{R}4_{mpkl} \equiv \int_0^1 X_{m,\xi\xi}X_p X_k X_l\, d\xi, \quad \overline{R}5_{mpkl} \equiv \int_0^1 X_{m,\xi\xi}X_{p,\xi}X_k X_l\, d\xi$$

$$\overline{R}6_{mpkl} \equiv \int_0^1 X_{m,\xi}X_{p,\xi}X_{k,\xi}X_l\, d\xi$$

$$\overline{R}7_{mpkl} \equiv \int_0^1 X_{m,\xi\xi}X_{p,\xi}X_{k,\xi}X_l\, d\xi$$

$$A1_{rs}^{mn} = \frac{1}{\pi^2}[-(\overline{A}_{11}m^2 + \overline{A}_{66}f^2n^2)\overline{A}_{mr}\overline{A}_{ns} + 2\overline{A}_{16}fmn\overline{B}_{rm}\overline{B}_{sn}] \quad \text{(A17)}$$

$$A2_{rs}^{mn} = \frac{1}{\pi^2}[-(\overline{A}_{16}m^2 + \overline{A}_{26}f^2n^2)\overline{A}_{mr}\overline{A}_{ns} + (\overline{A}_{12} + \overline{A}_{66})fmn\overline{B}_{rm}\overline{B}_{sn}] \quad \text{(A18)}$$

$$A3_{rs}^{mn} = \frac{1}{\pi^4}[-(\overline{B}_{11}\overline{E}4_{mr}\overline{E}1_{ns} - 3\overline{B}_{16}f\,\overline{E}3_{mr}\overline{E}2_{ns} - (\overline{B}_{12} + 2\overline{B}_{66})f^2\overline{E}2_{mr}\overline{E}3_{ns} - \overline{B}_{26}f^3\overline{E}1_{mr}\overline{E}4_{ns}] \quad \text{(A19)}$$

$$A4_{rs}^{mnpq} = \frac{\mathrm{h}}{\pi^4\mathrm{a}}[\overline{\mathrm{A}}_{11}\overline{\mathrm{G}}5_{mpr}\overline{G}1_{nqs} + \overline{A}_{16}f\overline{G}3_{mpr}\overline{G}2_{qns} + 2\overline{A}_{16}f\overline{G}4_{mpr}\overline{G}2_{nqs} + (\overline{A}_{12} + \overline{A}_{66})f^2\overline{G}2_{mpr}\overline{G}4_{nqs} + \overline{A}_{66}f^2\overline{G}2_{mpr}\overline{G}3_{qns} + \overline{A}_{26}f^3\overline{G}1_{mpr}\overline{G}5_{nqs}] \quad \text{(A20)}$$

$$B1_{rs}^{mn} = \frac{1}{\pi^2}[-(\overline{A}_{16}m^2 + \overline{A}_{26}f^2n^2)\overline{A}_{mr}\overline{A}_{ns} + (\overline{A}_{12} + \overline{A}_{66})fmn\overline{B}_{rm}\overline{B}_{sn}] \quad \text{(A21)}$$

$$B2_{rs}^{mn} = \frac{1}{\pi^2}[-(\overline{A}_{66}m^2 + \overline{A}_{22}f^2n^2)\overline{A}_{mr}\overline{A}_{ns} + 2\overline{A}_{26}fmn\overline{B}_{rm}\overline{B}_{sn}] \quad \text{(A22)}$$

$$B3_{rs}^{mn} = \frac{1}{\pi^4}[-\overline{B}_{16}\overline{E}4_{mr}\overline{E}1_{ns} - (\overline{B}_{12} + 2\overline{B}_{66})f\overline{E}3_{mr}\overline{E}2_{ns} - 3\overline{B}_{26}f^2\overline{E}2_{mr}\overline{E}3_{ns} - \overline{B}_{22}f^3\overline{E}1_{mr}\overline{E}4_{ns}] \quad \text{(A23)}$$

$$B4_{rs}^{mnpq} = \frac{\mathrm{h}}{\pi^4\mathrm{a}}[\overline{\mathrm{A}}_{16}\overline{\mathrm{G}}5_{mpr}\overline{G}1_{nqs} + \overline{A}_{66}f\overline{G}3_{mpr}\overline{G}2_{qns} + (\overline{A}_{12} + \overline{A}_{66})f\overline{G}4_{mpr}\overline{G}2_{nqs} + 2\overline{A}_{26}f^2\overline{G}2_{mpr}\overline{G}4_{nqs} + \overline{A}_{26}f^2\overline{G}2_{mpr}\overline{G}3_{qns} + \overline{A}_{22}f^3\overline{G}1_{mpr}\overline{G}5_{nqs}] \quad \text{(A24)}$$

$$C1_{rs}^{mn} = \frac{4}{\pi}\{[\overline{B}_{11}m^3 + (\overline{B}_{12} + 2\overline{B}_{66})f^2m^2]\overline{F}1\overline{E}1_{sn} + (3\overline{B}_{16}fm^2n + \overline{B}_{26}f^3n^3)\overline{E}1_{rm}\overline{F}1_{sn}\} \quad \text{(A25)}$$

$$C2_{rs}^{mn} = \frac{4}{\pi}\{[\overline{B}_{16}m^3 + 3\overline{B}_{26}f^2mn^2]\overline{F}1_{rm}\overline{E}1_{sn} + [(\overline{B}_{12} + 2\overline{B}_{66})fm^2n + \overline{B}_{22}f^3n^3]\overline{E}1_{rm}\overline{F}1_{sn}\} \quad \text{(A26)}$$

$$C3_{rs}^{mn} = \frac{4}{\pi^4}[\overline{P}5_{mr}\overline{P}1_{ns} + 4\overline{D}_{16}f\overline{P}4_{mr}\overline{P}2_{ns} + 2(\overline{D}_{12} + 2\overline{D}_{66})f^2\overline{P}3_{mr}\overline{P}3_{ns} + 4\overline{D}_{26}f^3\overline{P}2_{mr}\overline{P}4_{ns} + \overline{D}_{22}f^4\overline{P}1_{mr}\overline{P}5_{ns}] \quad \text{(A27)}$$

$$C4_{rs}^{mnpq} = -\frac{4h}{\pi^4a}[\overline{B}_{11}\overline{Q}7_{mpr}\overline{Q}1_{nqs} + \overline{B}_{16}f\overline{Q}6_{mpr}\overline{Q}2_{qns} + 3\overline{B}_{16}f\overline{Q}5_{mpr}\overline{Q}2_{nqs} + 2(\overline{B}_{12} - \overline{B}_{66})f^2\overline{Q}3_{mpr}\overline{Q}3_{nqs} + (\overline{B}_{12} + 2\overline{B}_{66})f^2\overline{Q}4_{mpr}\overline{Q}3_{nqs} + (\overline{B}_{12} + 2\overline{B}_{66})f^2\overline{Q}3_{mpr}\overline{Q}4_{nqs} - 2(\overline{B}_{12} - \overline{B}_{66})f^2\overline{Q}4_{mpr}\overline{Q}4_{qns} + 3\overline{B}_{26}f^3\overline{Q}2_{mpr}\overline{Q}5_{nqs} + \overline{B}_{26}f^3\overline{Q}2_{mpr}\overline{Q}6_{qns} + \overline{B}_{22}f^4\overline{Q}1_{mpr}\overline{Q}7_{qns}] \quad \text{(A28)}$$

$$C5_{rs}^{mnpq} = -\frac{4h}{\pi^3a}(\overline{A}_{11}m\overline{H}3_{prm}\overline{G}1_{qsn} + \overline{A}_{16}fn\overline{G}3_{prm}\overline{H}1_{qsn} + \overline{A}_{12}f^2m\overline{H}1_{prm}\overline{G}3_{qsn} + \overline{A}_{26}f^3n\overline{G}1_{prm}\overline{H}3_{qsn} + 2\overline{A}_{16}fm\overline{H}2_{prm}\overline{G}2_{qsn} + 2\overline{A}_{66}f^2n\overline{G}2_{prm}\overline{H}2_{qsn}) \quad \text{(A29)}$$

$$C6_{rs}^{mnpq} = -\frac{4h}{\pi^3a}(\overline{A}_{16}m\overline{H}3_{prm}\overline{G}1_{qsn} + \overline{A}_{12}fn\overline{G}3_{prm}\overline{H}1_{qsn} + \overline{A}_{26}f^2m\overline{H}1_{prm}\overline{G}3_{qsn} + \overline{A}_{22}f^3n\overline{G}1_{prm}\overline{H}3_{qsn} + 2\overline{A}_{66}fm\,\overline{H}2_{prm}\overline{G}2_{qsn} + 2\overline{A}_{26}f^2n\overline{G}2_{prm}\overline{H}2_{qsn}) \quad \text{(A30)}$$

$$C7_{rs}^{mnpqkl} = -\frac{4h^2}{\pi^4a^2}(\tfrac{1}{2}\overline{A}_{11}\overline{R}7_{mpkr}\overline{R}1_{nqls} + \tfrac{1}{2}\overline{A}_{12}f^2\overline{R}4_{mpkr}\overline{R}3_{qlns} + \overline{A}_{16}f\overline{R}5_{mpkr}\overline{R}2_{lnqs} + \tfrac{1}{2}\overline{A}_{12}f^2\overline{R}3_{pkmr}\overline{R}4_{nqls} + \tfrac{1}{2}\overline{A}_{22}f^4\overline{R}1_{mpkr}\overline{R}7_{nqls} + \overline{A}_{26}f^3\overline{R}2_{pmkr}\overline{R}5_{nlqs} + \overline{A}_{16}f\overline{R}6_{mpkr}\overline{R}2_{nqls} + \overline{A}_{26}f^3\overline{R}2_{mpkr}\overline{R}6_{nqls} + 2\overline{A}_{66}f^2\overline{R}3_{mpkr}\overline{R}3_{nlqs}) \quad \text{(A31)}$$

$$J_{rs}^{mn} = 4\overline{P}1_{mr}\overline{P}1_{ns} \quad \text{(A32)}$$

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